DIFFERENTIAL OPERATORS ON CURVES

by Thomas Bloom

The formalism given by Grothendieck [2, Ch. 16] for differential operators on schemes can be immediately applied to analytic spaces [1, 5]. The object of this note is to describe the germs of differential operators at a singular point on a complex curve at which the curve is irreducible.

§1. Let X be a complex curve irreducible at a singular point $p \in X$. Near p, the normalization of X is a 1-1 analytic map $f\colon \Delta \to X$ of the form $t \mapsto (f_1(t), \cdots, f_r(t))$. Here Δ is the unit disc in the plane and we assume $f(0) = p$, X is locally embedded in $\mathbf{C}^r$ and p is the origin of $\mathbf{C}^r$. The functions $\{f_i(t)\}_{i=1,\ldots,r}$ each vanish at $0 \in \Delta$, say to order n_i.

Furthermore, the subring R of germs at $0 \in \Delta$ of analytic functions which are induced by functions from X is of finite codimension (as a $\mathbf{C}$-vector space) in the ring of germs of analytic functions at $0 \in \Delta$.

We will denote by N the minimal integer with the property that if g vanishes at 0 to order $\geqq N$, then $g \in R$.

Differential operators on X lift to differential operators on Δ with meromorphic coefficients.

There is, in fact a natural 1-1 correspondence between germs at $p \in X$ of analytic differential operators on X and germs at $0 \in \Delta$ of differential operators with meromorphic coefficients which preserve R.

Now, let D be the germ at $0 \in \Delta$ of a differential operator with meromorphic coefficients (henceforth abbreviated m.d.o) and write D in the form

$$D = \sum a_{ij} t^j \frac{d^i}{dt^i} \quad \text{with } a_{ij} \in \mathbf{C}.$$

1.1 **Definition** [3]. The *strength* of D is defined as $\sup_{a_{ij} \neq 0}(i-j)$ and will be denoted by $\mathrm{str}(D)$. Thus $\mathrm{str}(D)$ is an integer, $\mathrm{str}(D_1 \circ D_2) = \mathrm{str}(D_1) + \mathrm{str}(D_2)$ and $\mathrm{str}(D_1 + D_2) \leqq \max(\mathrm{str}(D_1), \mathrm{str}(D_2))$.

1.2 **Remarks.** If $\mathrm{str}(D) \leqq -N$ then D preserves R.

We will say that D is of homogeneous strength if $i-j$ has a fixed value for all non-zero terms in the above expansion for D.

§2. To study differential operators at $p \in X$ we will study the equivalent problem of m.d.o 's which preserve R. We will first consider the case where $f_1(t), \cdots, f_r(t)$ are monomials in t. X is thus weighted homogeneous at p.

We let $S = \{m \in \mathbf{Z} \mid m = \sum_{i=1}^{r} n_i m_i$ where n_i is as above and the m_i are integers $\geqq 0\}$. The ring R thus consists of all convergent power series of the form $\sum_{s \in S} a_s t^s$ where $a_s \in \mathbf{C}$.

2.1. **Remarks.** Let D be a m.d.o which preserves R. Since $g \in R$ if and only if each monomial in the Taylor series for $g \in R$, each term in the expansion of D as a sum of operators of homogeneous strength preserves R.

For μ an integer we let $Z(\mu, R) = \{\alpha \mid \alpha$ is an integer > 0, $t^\alpha \in R$, but $t^{\alpha-\mu} \notin R\}$.

2.2. **Lemma.** *If* $\mu \in S$ *then* $\operatorname{card}(Z(\mu, R)) = \mu - 1$.
If $\mu \notin S$, $\mu > 0$ *then* $\operatorname{card}(Z(\mu, R)) \geqq \mu$.

Proof. Consider the congruence classes of integers $\operatorname{mod} \mu$. In each such congruence class $\not\equiv 0$ there is at least one $\alpha \in Z(\mu, R)$. If $\mu \in S$ there is, by the semi-group property of S, precisely one such α in each non-zero congruence class and none in the zero congruence class. Thus if $\mu \in S$, $\operatorname{card}(Z(\mu, R)) = \mu - 1$. If $\mu \notin S$ there is at least one $\alpha \in Z(\mu, R)$ with α in the zero congruence class. Thus if $\mu \notin S$, $\operatorname{card}(Z(\mu, R)) \geqq \mu$.

2.3. **Lemma.** *The minimal order of a m.d.o of homogeneous strength* μ *and no terms of order zero which preserves* R *is* $\operatorname{card}(Z(\mu, R)) + 1$. *Such an operator is unique up to a scalar multiple.*

Proof. An m.d.o of order k, homogeneous strength μ and no term of order zero can be written uniquely in the form

$$D = \sum_{i=1}^{k} a_i t^{i-\mu} \frac{d^i}{dt^i} \quad \text{with } a_i \in \mathbf{C}.$$

Now if D is to preserve R we must have $D(t^\alpha) = 0$ for all monomials $t^\alpha \in R$ for which $t^{\alpha-\mu} \notin R$, that is, for all $\alpha \in Z(\mu, R)$. These conditions impose certain linear relations on the coefficients a_i of the form

$$a_1\alpha + a_2\alpha(\alpha-1) + \cdots + a_k\alpha(\alpha-1)\cdots(\alpha-(k-1)) = 0.$$

If $k = \operatorname{card}(Z(\mu, R))$, the square matrix with rows

$$\alpha \ \alpha(\alpha-1) \cdots \alpha(\alpha-1)(\alpha-(k-1)), \ (\alpha \in Z(\mu, R))$$

is invertible [3, p. 41] since it is equivalent, by elementary column operations to the matrix with rows

$$\alpha \ \alpha^2 \cdots \alpha^k .$$

Thus, the minimal value of k which ensures a non-zero solution to the above linear homogeneous equations is $k = \operatorname{card}(Z(\mu, R)) + 1$.

2.4 **Corollary.** No m.d.o preserving R has a meromorphic leading coefficient when written as a sum of operators of decreasing order.

Proof. If D is an m.d.o of homogeneous strength $\mu > 0$ and has a meromorphic leading coefficient, then the order of D must be $< \mu$. This contradicts Lemmas 2.3 and 2.4. If $\mu \leqq 0$, then clearly no coefficient is meromorphic.

§3. We now turn to the case where $f_1(t), \cdots, f_r(t)$ are not necessarily monomials. We denote by $\bar{R}$ the ring formed by the initial terms in the Taylor expansion of elements in R. We denote by $\bar{S}$ the corresponding semi-group.

3.1. **Lemma.** *Let D be a m.d.o of strength μ which preserves R. Then D_μ, the terms in D of homogeneous strength μ, preserves $\bar{R}$.*

Proof. For $g \in R$ with initial term $\bar{g}$, the initial term of $D(g)$ is $D_\mu(\bar{g})$.

3.2 **Theorem.** *Let D be a m.d.o which preserves R and denote its order by* $\operatorname{ord}(D)$. *Then* $\operatorname{ord}(D) \geqq \operatorname{str}(D)$.

Proof. This follows from lemmas 2.2, 2.3, and 3.1.

3.3 **Theorem.** *Let P_μ be a m.d.o of homogeneous strength μ which preserves $\bar{R}$. A sufficient condition that there exists a m.d.o P' of strength $< \mu$ and order $\leqq m$ such that $P_\mu + P'$ preserves R is that* $\operatorname{card}(Z(j, \bar{R})) < m$ *for $j = \mu - 1, \mu - 2, \cdots, -N + 1$.*

Proof. As a preliminary step choose a **C**-basis for $R \bmod t^N$ as follows: The basis consists of polynomials $v_1, \cdots, v_l$ with initial terms $\bar{v}_1, \cdots, \bar{v}_l$ and such that the monomial $\bar{v}_i$ does not occur in the expression for v_j $(j \neq i)$. Given a convergent power series h, let $H =$ sum of the monomials in the expansion of h which are in $\bar{R} \bmod t^N$.

Say $H = \sum_{i=1}^{l} c_i \bar{v}_i$. Then $h \in R$ if and only if $h = \sum_{i=1}^{l} c_i v_i \pmod{t^N}$.

We will construct the required operator P' inductively as follows. Having chosen operators of homogeneous strength $P_{\mu+1}, \cdots, P_{r+1}$ choose P_r so that it satisfies the following conditions: For each $\alpha \in Z(r, \bar{R})$ take $g \in R$ with its initial term $\bar{g}$ a monomial of degree α. Let G = sum of the monomials in $(P_\mu +, \cdots, P_{r+1})(g)$ which are also in $\bar{R} \bmod t^N$ and are of order $< \alpha - r$.

Say $G = \sum_{i=1}^{l} d_i \bar{v}_i$. Let $\lambda t^{\alpha - r}$ be the monomial of order $\alpha - r$ in $\sum_{i=1}^{l} d_i v_i$ (possibly $\lambda = 0$). P_r is chosen so that $P_r(g_1) + P_{r+1}(g_2) + \cdots$, $P_\mu(g_{r+1-\mu}) = \lambda t^{\alpha-r}$ where $g_1 + g_2 + \cdots$ is the Taylor expansion of g in terms of order α, $\alpha + 1, \cdots$ *etc*. Thus for each $\alpha \in Z(r, \bar{R})$ one linear relation is imposed on the coefficients of P_r. Since $\operatorname{card} Z(r, \bar{R}) < m$ for $\mu > r \geqq -N + 1$ by hypothesis, there is an operator of order $\leqq m$ and homogeneous strength r satisfying the above equations (as in lemma 2.3). The above procedure is repeated until $r = -N + 1$. Then $P' = \sum_{=\mu-1}^{-N+1} P_i$ has the required properties.

3.4. **Corollary.** There is a m.d.o of strength m and order m preserving R if m is sufficiently large.

Proof. Let $M = \sup_{N > j > -N+1}(\operatorname{card} Z(j, \bar{R}) + 1)$. For $j \geqq N$, $j \in \bar{S}$; so $\operatorname{card} Z(j, \bar{R}) = j - 1$. Now consider any $m \geqq M$. $M \geqq N$ so $m \in \bar{S}$. Thus, by Lemmas 2.2 and 2.3 there is a m.d.o of order m and strength m preserving $\bar{R}$. Now, $\operatorname{card} Z(j, \bar{R}) < m$ for $j = m - 1, \cdots, -N + 1$ so, applying Theorem 3.3, we obtain the required m.d.o.

Let A be an analytic ring with unique maximal ideal $\mathscr{M}$. Under composition, the differential operators on A form a non-commutative ring, denoted $\operatorname{Diff}(A)$ which is filtered by order. We denote by $\operatorname{Gr}\operatorname{Diff}(A)$ the associated graded ring which is commutative. Given $D \in \operatorname{Diff}(A)$ we denote by $\tilde{D}$ the homogeneous element it induces in $\operatorname{Gr}\operatorname{Diff}(A)$.

3.5. **Corollary.** Given a differential operator $D \in \operatorname{Diff}(R)$, there exists an integer s such that $\tilde{D}^s \in \mathscr{M} \operatorname{Gr}\operatorname{Diff}(R)$ if and only if $\operatorname{ord}(D) > \operatorname{str}(D)$.

Proof. Let D be of strength μ and order $> \mu$. D may be written in the form $D = \sum_{i=0}^{k} h_i (d^i/dt^i)$ where h_k is analytic and $h_k(0) = 0$. Now D^M (M is the integer introduced in Cor. 3.4) is an operator with leading term $(h_k)^M (d^{kM}/dt^{kM})$ and there is, by Corollary 3.4, an m.d.o P preserving R of order kM and strength kM. Now $h_k^M \in \mathscr{M}$, so there exists an element $h \in \mathscr{M}$ such that $\operatorname{ord}(D^M - hP) < kM$. Thus $\tilde{D}^M = h\tilde{P}$ in $\operatorname{Gr}\operatorname{Diff}(A)$.

Conversely, if $\tilde{D}^s \in \mathscr{M} \operatorname{Gr}\operatorname{Diff}(A)$, there is an m.d.o P with $\operatorname{ord}(P) = \operatorname{ord}(D^s) = s\operatorname{ord}(D)$ such that $\operatorname{ord}(D^s - hP) < \operatorname{ord}(P)$ for some $h \in \mathscr{M}$. Say $D^s = hP + P'$ where $\operatorname{ord}(P') < \operatorname{ord}(D^s)$. Now $\operatorname{str}(D^s) \leqq \max(\operatorname{str}(hP), \operatorname{str}(P'))$. But $\operatorname{str}(hP) < \operatorname{str}(P) \leqq \operatorname{ord}(P) = \operatorname{ord}(D^s)$.

$$\operatorname{str}(P') \leqq \operatorname{ord}(P') < \operatorname{ord}(D^s).$$

In either case, $\operatorname{str}(D^s) < \operatorname{ord}(D^s)$, so

$$\operatorname{str}(D) < \operatorname{ord}(D).$$

Thus, we may associate to the ring R a semi-group $S' = \{\mu \in \mathbf{Z} \mid \text{there exists a m.d.o } D \text{ preserving } R \text{ and such that } \operatorname{ord}(D) = \operatorname{str}(D) = \mu\}$. S' is, of course, intrinsic to the singularity of X at p since the above corollary gives an intrinsic characterization of those operators D such that $\operatorname{ord}(D) = \operatorname{str}(D)$. In fact, S' is the semi-group of integers which are the degrees of the homogeneous elements of the $\mathbf{C}$-algebra $\operatorname{GrDiff}(R)/\sqrt{\mathscr{M}\operatorname{GrDiff}(R)}$. Now $\operatorname{Gr}\operatorname{Diff}(R)/\sqrt{\mathscr{M}\operatorname{Gr}\operatorname{Diff}(R)}$ is a one-dimensional Noetherian ring.

The relations in the ring may be described as follows: Let a and b be homogeneous elements of degrees α, β respectively. Let $\gamma = g.c.d.(\alpha, \beta)$. Then there exists non-zero $\lambda_1, \lambda_2 \in \mathbf{C}$ such that

$$\lambda_1 a^{\alpha/\gamma} = \lambda_2 b^{\beta/\gamma}.$$

and all relations are generated by ones of the above form. Thus the ring is completely characterized by S'.

For the weighted homogeneous curves of section 2, $S' = S$; but, as example 4.2 shows, $S' \neq \bar{S}$ in general.

3.6 **Corollary.** $\operatorname{Gr}\operatorname{Diff}(R)$ and $\operatorname{Diff}(R)$ are finitely generated R-algebras.

Proof. It is a standard algebraic argument that if $\operatorname{Gr}\operatorname{Diff}(R)$ is finitely generated then $\operatorname{Diff}(R)$ is finitely generated as a left or right R-algebra. We will prove that $\operatorname{Gr}\operatorname{Diff}(R)$ is finitely generated, in fact, generated by operators of order $< 2M$ where M is the integer introduced in corollary 3.4.

Since $\operatorname{card}(Z(j, \bar{R})) < M$ for $j = M, M-1, \cdots, M-(N-1)$, there exist (following the procedure of Lemma 2.3) m.d.o's $P_0, \cdots, P_{N-1}$ preserving $\bar{R}$, of order M and such that P_i is of homogeneous strength $M-i$ for $i = 0, \cdots, N-1$. Applying Theorem 3.3 there exist m.d.o's $Q_0, \cdots, Q_{N-1}$ preserving R, of order M and such that $\operatorname{str}(Q_i) = M - i$ for $i = 0, \cdots, N-1$.

Now, let $T \in \operatorname{Diff}(R)$ be of order $\geqq 2M$ and suppose $\operatorname{ord}(T) - \operatorname{str}(T) \equiv K \bmod N$. Then, take (by 3.4) T' a m.d.o preserving R such that $\operatorname{ord}(T') = \operatorname{str}(T') = \operatorname{ord}(T) - M$. Consider $Q_K \circ T'$. Now $\operatorname{ord}(Q_K \circ T') = \operatorname{ord}(T)$ and $\operatorname{str}(T - Q_K \circ T') \equiv 0 \bmod N$. Thus there exists $h \in R$ such that $\operatorname{ord}(T - hQ_K \circ T') < \operatorname{ord}(T)$ so that $h\tilde{Q}_K\tilde{T}' = \tilde{T}$ in $\operatorname{Gr}\operatorname{Diff}(R)$.

If $\operatorname{ord}(T') \geqq 2M$, we repeat the above procedure. It is clear, thus, that operators of order $< 2M$ generate $\operatorname{Gr}\operatorname{Diff}(R)$ as an R-algebra.

4. We will illustrate by two examples.

4.1 $f_1(t) = t^2, f_2(t) = t^3$. Here $R = \{ \sum_{i=0}^{\infty} a_i t^i \,|\, a_1 = 0\}$. $X = \{x, y \in \mathbf{C}^2 \,|\, x^3 = y^2\}$.
The operators

$$D_1 = \frac{d^2}{dt^2} - \frac{2}{t}\frac{d}{dt} \quad \text{and} \quad D_2 = \frac{d^3}{dt^3} - \frac{3}{t}\frac{d^2}{dt^2} + \frac{3}{t^2}\frac{d}{dt}$$

preserve R. They represent tangent vectors which are a basis for the Zariski tangent space to X at $(0,0)$.

Since $\operatorname{ord}(D_1) = \operatorname{str}(D_1) = 2$, and $\operatorname{ord}(D_2) = \operatorname{str}(D_2) = 3$, they generate

$$\frac{\operatorname{Gr}\operatorname{Diff}(R)}{\sqrt{\mathscr{M}\operatorname{Gr}\operatorname{Diff}(R)}}.$$

4.2. $f_1(t) = t^4$, $f_2(t) = t^5 + t^6$. Here $N = 12$.

As a special $\mathbf{C}$ basis of $R \bmod t^{12}$ (as in Lemma 3.3) we have $v_1 = t^4$, $v_2 = t^5 + t^6$, $v_3 = t^8$, $v_4 = t^9 - 2t^{11}$, $v_5 = t^{10} + 2t^{11}$. $\bar{R}$ is the ring associated to t^4, t^5.

There is no m.d.o preserving R of strength 4 and order 4. For suppose P were such a m.d.o and $P = P_4 + P_3 + , \cdots$ its expansion into terms of homogeneous strength. P_4 would preserve $\bar{R}$ and would be unique up to a constant multiple. P_3 must satisfy the equations

$$\begin{aligned} P_3(t^4) &= 0 \\ P_3(t^5) + P_4(t^6) &= 0 \\ tP_4(t^9) &= P_3(t^9) \\ 2P_4(t^{11}) + P_3(t^{10}) &= 0 \\ tP_4(t^{14}) &= 2P_3(t^{14}), \end{aligned}$$

These are five linearly independent equations for the coefficients of P_3, so it must be of order $\geqq 5$.

In fact, the minimal order operator P such that P preserves R and $P(t^4)$ is nonzero at zero is of order 5. Thus P is an operator of minimal order which represents a tangent vector; however, $\operatorname{ord}(P) > \operatorname{str}(P)$.

4.3 Kantor [*4*] and Stutz [*6*] studied differential operators on certain analytic spaces of dimension > 1. A detailed study of differential operators on the curves $x^b - y^a = 0$ was made by Jaffe [*3*]. With regard to the specific questions discussed in this note we point out that if A is the local ring at $(0,0,0)$ of analytic functions on the $\{x, y, z \in \mathbf{C}^3 \mid xy^2 - z^2 = 0\}$, then $\operatorname{Gr}\operatorname{Diff}(A)$ is not a finitely generated A-algebra whereas $\operatorname{Diff}(A)$ is a finitely generated algebra.

Note: It has been brought to the author's notice that similar results to the ones in this note, in particular Corollary 3.6, have been announced by Jean-Pierre Vigué, *C. R. Acad. Sc. Paris*, t. **274** (March 1972), 895–899.

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